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Strings on the deformed $T^{1,1}$: giant magnon and single spike solutions

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ABSTRACT: In this paper we have found giant magnon and single spike string solutions in a sector of the gamma-deformed conifold. We examined the dispersion relations and find a behavior analogous to the undeformed case. The transcendental functional relations between the conserved charges are shifted by a certain gamma-dependent term. The latter is proportional to the total momentum and thus qualitatively different from known cases.

KEYWORDS: AdS-CFT Correspondence, Conformal Field Models in String Theory

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1 Introduction

One of the most exciting topics in the high energy theory over the last decades is the correspondence between the strings and the gauge fields. One of the most promising explicit realizations of this correspondence was provided by the Maldacena conjecture about AdS/CFT correspondence [1].

The semi-classical string has played an important role in studying various aspects of the AdS_5/SYM_4 correspondence [2]–[41]. The developments and successes in this particular case suggest the methods and tools that should be used to investigate the new emergent duality. The best studied example of the duality between the string and gauge theories is the AdS/CFT correspondence on $AdS_5 \times S^5$. One of the most important predictions of the correspondence is the equivalence between the spectrum of the string theory and the spectrum of anomalous dimensions of gauge invariant operators. There has been a good deal of success recently in comparing of the energies of the semiclassical strings and the anomalous dimensions of the gauge theory operators. Some of the lessons we have learned from the AdS/CFT correspondence on $AdS_5 \times S^5$ teach us the following. The strong evidences for integrability suggest to look for an appropriate scattering matrix S

which in principle encodes in a certain way the dynamics. On the string side, in the strong coupling limit the S matrix can be interpreted as describing the two-body scattering of elementary excitations on the world sheet. When their world-sheet momenta become large, these excitations can be described as special types of solitonic solutions, or giant magnons, and the interpolating region is described by the dynamics of the so-called near-flat-space regime [12, 28, 29]. On the gauge theory side, the action of the dilatation operator on single-trace gauge-invariant operators is the same as that of a Hamiltonian acting on the states of a certain spin chain [5]. This turns out to be of great advantage because one can diagonalize the matrix of anomalous dimensions by using the “magic” algebraic Bethe ansatz technique. The insertion of different operators into the single trace long operators is interpreted as magnons and the S-matrix factorizes to two-magnon scatterings governing the spectrum.

On the string theory side, the corresponding classical string solutions are called giant magnons and have the shape of arcs moving along some isometry direction. The angle deficit defined by the end points of the arcs is identified as the momentum of the magnon, while the dispersion relations determine the anomalous dimension of a certain gauge theory operator at strong coupling.

Another important string solution is the so called single spike string. Its shape has only a single spike and a large winding number in some isometry direction. While the giant magnon solutions can be interpreted as higher twist operators in the field theory, the single spike solutions do not seem to be directly related to particular field theory operators. However, in [14] an interpretation of this solution as a spin chain Hubbard model, which means the antiferromagnetic phase of the corresponding spin chain, was suggested, but the relation to the field theory operators is still unclear. Although not completely understood, the spiky string solutions are believed to play an important role in the AdS/CFT correspondence.

After the impressive achievements in the most supersymmetric example of the AdS/CFT correspondence, namely $AdS_5 \times S^5$, it is important to extend the considerations to less supersymmetric gauge theories, moreover that the latter are more interesting from the physical point of view. In the known to the authors cases of less supersymmetry we are not that lucky to have firm evidences for integrability as in the $AdS_5 \times S^5$ case. The lessons from the most supersymmetric case, however, suggest investigation of the string solutions of the giant magnon and the single spike type. These solitonic solutions are supposed to play an analogous role, namely, their quantum numbers to be related to the corresponding gauge theory quantum numbers in a way dictated by the holographic correspondence. Thus, although we have no clear signs for integrability one can still concentrate on these sectors and analyze the information that can be extracted on both sides of the correspondence.

There are several ways to find a theory with less supersymmetry. One of them is suggested by Lunin and Maldacena [30]. The authors consider $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) and its marginal deformations [31]. In [30] the supergravity dual of marginally deformed supersymmetric Yang-Mills theory has been identified. An explicit deforming procedure, called TsT transformation, and the integrability of the resulting backgrounds

has been presented in [32]. Giant magnons and single spike strings have been studied in [33]–[37]. In [33, 34] the deformation parameter enters the dispersion relation for the giant magnon as a shift by $\pi\gamma$. In [37] it was argued that, in the limit of the conserved charge $\mathcal{J} = J/g \rightarrow \infty$ and upon the identification $\gamma\sqrt{\lambda} \sim \hat{\gamma}$, the shift by γ should not be seen by the classical theory. It is important however that the deformation introduces a non-trivial twist in the boundary conditions for the isometry directions with essential consequences.

There is another way to approach less supersymmetric backgrounds. The experience from the AdS/CFT correspondence suggests that one can take a stack of N D3 branes and place them not in a flat space, but at the apex of a conifold [42]. This model possess a lot of interesting features and allows to build gauge theory operators of great physical importance. The resulting ten dimensional space time takes form of the direct product $AdS_5 \times T^{1,1}$. Since then infinite families of five dimensional spaces, called Sasaki-Einstein spaces, complementing AdS_5 space have been constructed [43]–[44] as well as their gauge theory duals were identified [45]–[48]. Further developments can be traced in [43]–[54].

The powerful solution generating technique based on the Lunin-Maldacena construction has been applied to various backgrounds in [30, 38]. The deformations in these papers also include the conifold which is certainly of interest for the AdS/CFT correspondence.

Inspired by the considerations in [54] and [55, 57], we investigate the giant magnon and the single spike string solutions in the beta-deformed conifold background. The dispersion relations are supposed to describe the anomalous dimensions of a particular class of gauge theory operators. We expect our results to shed some light on the conjectured duality.

The paper is organized as follows. In the second section we review the result of [54], beta-deformations and giant magnons and spiky strings in such backgrounds. Section 3 presents the string theory in a consistently truncated subsector of $T^{1,1}$. In section 4 we derive the dispersion relations for the cases of giant magnon and single spike string solutions. The obtained results are summarized in Conclusions. Some helpful formulae are presented in an appendix.

2 Review of the known results

In this section we review first the giant magnon and the single spike strings on the conifold and then briefly describe the same issues in the beta-deformed S_γ^5 (here $\gamma = \Re\beta$). We fix here some notations and present the methods which will be used in what follows.

2.1 Giant magnons and single spike strings on the conifold

Here we will briefly review the results of [54]. The metric of the conifold can be written as $dr^2 + r^2 d\Omega_{T^{1,1}}^2$ and combined with the metric of a stack of N D3 branes can be organized as $AdS_5 \times T^{1,1}$. The description best suited for our purpose is as follows. Let us consider strings moving in $T^{1,1}$, which is a homogeneous space $(SU(2) \times SU(2))/U(1)$, with $U(1)$

chosen to be a diagonal subgroup of the maximal torus in $SU(2) \times SU(2)$. One can start with more general set up of squashed spheres employed in [54] with an explicit form of the metric written as $U(1)$ bundle over $S^2 \times S^2$,

$$ds^2 = a (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + b (d\psi + p \cos \theta_1 d\phi_1 + q \cos \theta_2 d\phi_2)^2. \quad (2.1)$$

Here θ_i, ϕ_i are the coordinates of the two S^2 's, and the $U(1)$ fiber is parameterized by $\psi \in [0, 4\pi]$. The space is an Einstein manifold if the following choice of the parameters is made $a = \frac{1}{6}, b = \frac{1}{9}$. Supersymmetry requirements further restricts $p = q = 1$ and thus the space becomes supersymmetric, i.e. the resulting Sasaki-Einstein manifold allows two Killing spinors, hence $\mathcal{N} = 1$ supersymmetry.

The part $T^{1,1}$ provides the angular part of a singular Calabi-Yau manifold. One can easily see from eq. (2.1) that the isometry is $SU(2) \times SU(2) \times U(1)$. The three mutually commuting Killing vectors can be chosen as $\partial_{\phi_1}, \partial_{\phi_2}, \partial_{\psi}$.

We will proceed however with the choice $p = q = 1$ but with squashing parameter b unfixed ($a = b/4$). It was shown in [54] that one can consistently set, say $\theta_2, \phi_2 = const$. The starting point then is the subspace of $T^{1,1}$ corresponding to the metric

$$ds^2 = -dt^2 + \frac{b}{4} [d\theta^2 + \sin^2 \theta d\phi^2 + b(d\psi - \cos \theta d\phi)^2], \quad (2.2)$$

where the time coordinate $t \in \mathbb{R}$ originates from AdS_5 .

Equations. Let us consider the sector defined by $\theta_2, \phi_2 = const$. To obtain solitonic solutions we use the ansatz

$$t = \kappa\tau, \theta \equiv \theta_1 = \theta(y), \Psi = \omega_\psi\tau + \psi(y), \Phi = \omega_\phi\tau + \phi(y), \quad (2.3)$$

where $y = -d\tau + c\sigma$, Ψ describes the $U(1)$ fiber and $\Phi \equiv \phi_1$.

Integrating once the equations for the angles Ψ and Φ in terms of θ and using the Virasoro constraints one finds

$$u' = 4[a_4u^4 + a_3u^3 + a_2u^2 + a_1u + a_0]. \quad (2.4)$$

where $u = \cos^2 \theta/2$. Imposing appropriate boundary conditions we end up with ($\alpha_{\pm} > 0$).

$$u'^2 = \alpha^2 \omega_\phi u^2 (\alpha_+ - u)(u + \alpha_-). \quad (2.5)$$

The following relations between the integration constants and the frequencies determine the profile of the solution ($A_\psi = A_\phi$)

$$A_\phi = \frac{d}{9} (\omega_\psi + \omega_\phi) \quad \text{giant magnon} \quad (2.6)$$

$$A_\phi = \frac{c^2}{9d^2} (\omega_\psi + \omega_\phi) \quad \text{single spike.} \quad (2.7)$$

Dispersion relations. For the magnon type and the spiky string solutions the conserved charges are

$$P_t = -T \frac{b}{2} (\omega_\psi + \omega_\phi) \tag{2.8}$$

$$P_\psi = Tb \left(\frac{b}{4} (\omega_\psi + \omega_\phi) - \frac{b\omega_\phi}{2(1-d/c)^2} u(y) \right) \tag{2.9}$$

$$P_\phi = Tb \left(\frac{b}{4} (\omega_\psi + \omega_\phi) - \frac{b\omega_\phi}{2(1-(d/c)^2)} \left(2(1-b)u(y)^2 + (b\Omega - 2(1-b))u(y) \right) \right), \tag{2.10}$$

where $u(y) = \cos^2 \theta/2$ and $\Omega = (1-b)/b$.

The finite quantities giving the dispersion relations are

$$\mathcal{E} - \frac{2}{b} \mathcal{J}_\psi, \quad \mathcal{E} - \frac{2}{b} \mathcal{J}_\phi, \quad \mathcal{E} - \frac{\mathcal{J}_\psi + \mathcal{J}_\phi}{b}, \quad \frac{\mathcal{J}_\psi - \mathcal{J}_\phi}{b}. \tag{2.11}$$

The *giant magnon* dispersion relation on the conifold is

$$\frac{\sqrt{3}}{2} (\mathcal{E} - 3\mathcal{J}_\psi) = \frac{\sqrt{3}(\mathcal{E} - 3\mathcal{J}_\psi)/2 - \cos \Delta\phi}{\sin(\sqrt{3}(\mathcal{E} - 3\mathcal{J}_\psi)/2)}. \tag{2.12}$$

Note that the dispersion relations are quite different from those in the most supersymmetric case.

The *single spike* string solutions obey the following dispersion relations

$$\frac{3\sqrt{3}J_\psi}{2} = \frac{\cos(3\sqrt{3}J_\psi) - \cos(2/3E - \Delta\varphi)}{\sin(3\sqrt{3}J_\psi)}. \tag{2.13}$$

Again the dispersion relations are quite different from those in the most supersymmetric case, namely they have transcendental functional dependence between the charges.

Gauge theory side. The dual conformal field theory is known as the Klebanov-Witten model [42] and is constructed considering a stack of D3 branes, placed at the tip of a conifold.

The dual conformal field theory is identified as $\mathcal{N} = 1$ supersymmetric $U(N) \times U(N)$ gauge theory with two chiral multiplets A_i in (N, \overline{N}) and another two, usually denoted by B_i , in (\overline{N}, N) . The angular part of the conifold is $T^{1,1}$ and its isometries determine the global symmetries of the gauge theory. Being $U(1)$ bundle over $S^2 \times S^2$, this theory obviously has $SU(2) \times SU(2)$ global symmetry which act separately on the doublets A_i, B_i , and also a non-anomalous $U(1)$ R-symmetry.

The most general superpotential which respects the $SU(2) \times SU(2) \times U(1)_R$ symmetry is a quartic superpotential of the form

$$W = \frac{g}{2} \epsilon^{ij} \epsilon^{kl} \text{Tr} A_i B_k A_j B_l. \tag{2.14}$$

Note that there is also a \mathbf{Z}_2 symmetry. In the geometric picture, i.e. on the conifold, it acts as a reflection and from the gauge theory point of view it exchanges the two pairs A_i and B_j .

The AdS/CFT correspondence suggests that the anomalous dimension of the gauge theory operators are encoded in the dispersion relation in the string theory. Therefore, here we are interested primarily in the conserved quantities which are the energy $E = \sqrt{\lambda}\kappa$ and the following three angular momenta,

$$J_A \equiv P_{\phi_1}, \quad J_B \equiv P_{\phi_2}, \quad J_R \equiv P_\psi. \quad (2.15)$$

In order to have a reliable comparison we must consider the long composite operators constructed out of A_i and B_j . Then, it is natural to suggest a correspondence between quantum numbers in the string theory and the dual operators. As it was shown in [42], the strings moving in $T^{1,1}$ are dual to pure scalar operators, i.e. they do not contain fermions, covariant derivatives or gauge field strengths. One can construct a scalar by making use of the fact that they are in the bi-fundamental representation. Therefore, the gauge singlets have the form

$$\text{Tr}\left(A B \cdots A \bar{A} \cdots \bar{B} B \cdots \bar{B} \bar{A} \cdots\right). \quad (2.16)$$

This form of the operators suggests the correspondence

$$J_A \longleftrightarrow \frac{1}{2} \left[\#(A_1) - \#(A_2) + \#(\bar{A}_2) - \#(\bar{A}_1) \right] \quad (2.17)$$

$$J_B \longleftrightarrow \frac{1}{2} \left[\#(B_1) - \#(B_2) + \#(\bar{B}_2) - \#(\bar{B}_1) \right] \quad (2.18)$$

$$J_R \longleftrightarrow \frac{1}{4} \left[\#(A_i) + \#(B_i) - \#(\bar{A}_i) - \#(\bar{B}_i) \right] \quad (2.19)$$

where $\#(A_1)$ is the number of A_1 's under the trace of the dual composite operator etc.

We note that there exists an inequality between the bare dimension and the R -charge, which is quite natural when written in terms of string variables,

$$E \geq 3|J_R|. \quad (2.20)$$

On the gauge theory side it comes from the unitarity bound of $\mathcal{N} = 1$ superconformal algebra. When the bound is saturated the primary fields close a chiral ring. The complete dictionary between conserved charges in the string theory and the dual gauge theory operator remains an open problem.

The derivation of the general string solution is a subject to much more complicated task related to issues as integrability etc.

2.2 Giant magnons and single spike strings on S^3_γ

Here we review the β -deformed $AdS_5 \times S^5$ background found by Lunin and Maldacena [30]. This background is conjectured to be dual to the Leigh-Strassler marginal deformations of

$\mathcal{N} = 4$ SYM [31]. We note that this background can be obtained from pure $AdS_5 \times S^5$ by a series of TsT transformations as described in [32]. The deformation parameter $\beta = \gamma + i\sigma_d$ is in general a complex number, but in our analysis we will consider $\sigma_d = 0$, in this case the deformation is called γ -deformation. The resulting supergravity background dual to real β -deformations of $\mathcal{N} = 4$ SYM is:

$$ds^2 = R^2 \left(ds^2_{AdS_5} + \sum_{i=1}^3 (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \tilde{\gamma}^2 G\mu_1^2 \mu_2^2 \mu_3^2 \left(\sum_{i=1}^3 d\phi_i^2 \right) \right) \quad (2.21)$$

This background includes also a dilaton field as well as RR and NS-NS form fields. The relevant form for our classical string analysis will be the antisymmetric B-field:

$$B = R^2 \tilde{\gamma} G (\mu_1^2 \mu_2^2 d\phi_1 d\phi_2 + \mu_2^2 \mu_3^2 d\phi_2 d\phi_3 + \mu_1^2 \mu_3^2 d\phi_1 d\phi_3) \quad (2.22)$$

In the above formulae we have defined

$$\begin{aligned} \tilde{\gamma} &= R^2 \gamma & R^2 &= \sqrt{4\pi g_s N} = \sqrt{\lambda} \\ G &= \frac{1}{1 + \tilde{\gamma}^2 (\mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_1^2 \mu_3^2)} \end{aligned} \quad (2.23)$$

$$\mu_1 = \sin \theta \cos \psi \quad \mu_2 = \cos \theta \quad \mu_3 = \sin \theta \sin \psi$$

Where $(\theta, \psi, \phi_1, \phi_2, \phi_3)$ are the usual S^5 variables. This is a deformation of the $AdS_5 \times S^5$ background governed by a single real deformation parameter $\tilde{\gamma}$ and thus provides a useful setting for the extension of the classical strings/spin chain/gauge theory duality to less supersymmetric cases.

Let us consider the motion of a rigid string on S_γ^3 . This space can be thought of as a subspace of the γ -deformation of $AdS_5 \times S^5$ presented above

$$\mu_3 = 0, \quad \phi_3 = 0 \quad \text{i.e.} \quad \psi = 0, \quad \phi_3 = 0. \quad (2.24)$$

The relevant part of the γ -deformed $AdS_5 \times S^5$ is

$$ds^2 = -dt^2 + d\theta^2 + G \sin^2 \theta d\phi_1^2 + G \cos^2 \theta d\phi_2^2 \quad (2.25)$$

where $G = \frac{1}{1 + \tilde{\gamma}^2 \sin^2 \theta \cos^2 \theta}$ and due to the series of T-dualities there is a non-zero component of the B-field

$$B_{\phi_1 \phi_2} = \tilde{\gamma} G \sin^2 \theta \cos^2 \theta \quad (2.26)$$

We will work in the conformal gauge and thus use the Polyakov action ($T = \frac{\sqrt{\lambda}}{2\pi}$)

$$\begin{aligned} S = \frac{T}{2} \int d^2\sigma \left[-(\partial_\tau t)^2 + (\partial_\tau \theta)^2 - (\partial_\sigma \theta)^2 + G \sin^2 \theta ((\partial_\tau \phi_1)^2 - (\partial_\sigma \phi_1)^2) \right. \\ \left. + G \cos^2 \theta ((\partial_\tau \phi_2)^2 - (\partial_\sigma \phi_2)^2) + 2\gamma G \sin^2 \theta \cos^2 \theta (\partial_\tau \phi_1 \partial_\sigma \phi_2 - \partial_\sigma \phi_1 \partial_\tau \phi_2) \right] \end{aligned} \quad (2.27)$$

which is supplemented by the Virasoro constraints

$$g_{\mu\nu}\partial_\tau X^\mu\partial_\sigma X^\nu = 0 \qquad g_{\mu\nu}(\partial_\tau X^\mu\partial_\tau X^\nu + \partial_\sigma X^\mu\partial_\sigma X^\nu) = 0. \quad (2.28)$$

Here $g_{\mu\nu}$ is the metric (2.25) and $X^\mu = \{t, \theta, \phi_1, \phi_2\}$. The ansatz

$$t = \kappa\tau \qquad \theta = \theta(y) \qquad \phi_1 = \omega_1\tau + \tilde{\phi}_1(y) \qquad \phi_2 = \omega_2\tau + \tilde{\phi}_2(y) \quad (2.29)$$

describes the motion of rigid strings on the deformed 3-sphere, here we have defined a new variable $y = \alpha\sigma + \beta\tau$. One can substitute the above ansatz in the equations of motion and use one of the Virasoro constraints to find three first order differential equations for the unknown functions:

$$\begin{aligned} \tilde{\phi}_1' &= \frac{1}{\alpha^2 - \beta^2} \left(\frac{A}{G \sin^2 \theta} + \beta\omega_1 - \tilde{\gamma}\alpha\omega_2 \cos^2 \theta \right) \\ \tilde{\phi}_2' &= \frac{1}{\alpha^2 - \beta^2} \left(\frac{B}{G \cos^2 \theta} + \beta\omega_2 + \tilde{\gamma}\alpha\omega_1 \sin^2 \theta \right) \\ (\theta')^2 &= \frac{1}{(\alpha^2 - \beta^2)^2} \left[(\alpha^2 + \beta^2)\kappa^2 - \frac{A^2}{G \sin^2 \theta} - \frac{B^2}{G \cos^2 \theta} - \alpha^2\omega_1^2 \sin^2 \theta - \alpha^2\omega_2^2 \cos^2 \theta \right. \\ &\quad \left. + 2\tilde{\gamma}\alpha(\omega_2 A \cos^2 \theta - \omega_1 B \sin^2 \theta) \right] \end{aligned} \quad (2.30)$$

A and B are integration constants and the prime denotes derivative with respect to y . The other Virasoro constraints provides the following relation between the parameters

$$A\omega_1 + B\omega_2 + \beta\kappa^2 = 0 \quad (2.31)$$

This system has three conserved quantities - the energy and two angular momenta:

$$\begin{aligned} E &= 2T \frac{\kappa}{\alpha} \int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta'} \\ J_1 &= 2 \frac{T}{\alpha} \int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta'} G \sin^2 \theta \left[\omega_1 + \beta\tilde{\phi}_1' + \tilde{\gamma}\alpha \cos^2 \theta \tilde{\phi}_2' \right] \\ J_2 &= 2 \frac{T}{\alpha} \int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta'} G \cos^2 \theta \left[\omega_2 + \beta\tilde{\phi}_2' + \tilde{\gamma}\alpha \sin^2 \theta \tilde{\phi}_1' \right] \end{aligned} \quad (2.32)$$

where the integration is performed over the range of the coordinate θ . In the analysis below we will find solutions of the above equations and relations between the energy and the angular momenta for some special values of the parameters. These solutions include the giant magnon and the single spike solution on the deformed S^3 .

The conditions which determine the type of the solution come from the requirement of existence of a turning point at $\theta = \pi/2$. This condition sets $B = 0$ and provides the following choice

$$\begin{aligned} (i) \quad & \frac{\kappa^2}{\omega_1^2} = 1 \qquad \text{the giant magnon solution of [12]} \\ (ii) \quad & \frac{\kappa^2 \beta^2}{\alpha^2 \omega_1^2} = 1 \qquad \text{the single spike solution of [14]} \end{aligned} \quad (2.33)$$

The dispersion relations in the two cases are as follows.

Giant magnons. If we choose $\kappa^2 = \omega_1^2$ (which through the Virasoro constraint implies $A = -\beta\omega_1$) we get the giant magnon solution on S_γ^3 found in [34, 35]. The equations of motion for this case are:

$$\begin{aligned}\tilde{\phi}'_1 &= -\frac{\cos^2 \theta}{\alpha^2 - \beta^2} \left(\frac{\beta\omega_1}{\sin^2 \theta} + \tilde{\gamma}\alpha\omega_2 + \tilde{\gamma}^2\beta\omega_1 \right) \\ \tilde{\phi}'_2 &= \frac{\beta\omega_2 + \tilde{\gamma}\alpha\omega_1 \sin^2 \theta}{\alpha^2 - \beta^2} \\ \theta' &= \frac{\alpha\Omega_0}{(\alpha^2 - \beta^2)} \frac{\cos \theta}{\sin \theta} \sqrt{\sin^2 \theta - \sin^2 \theta_0}\end{aligned}\tag{2.34}$$

where we have defined

$$\sin \theta_0 = \frac{\beta\omega_1}{\alpha\Omega_0} \quad \text{and} \quad \Omega_0 = \sqrt{\omega_1^2 - \left(\omega_2 + \tilde{\gamma} \frac{\beta\omega_1}{\alpha} \right)^2}\tag{2.35}$$

Using the expressions for the energy and the angular momentum (2.32) and equations (2.34) we find

$$\begin{aligned}E - J_1 &= 2T \frac{\omega_1}{\Omega_0} \cos \theta_0 \\ J_2 &= 2T \left(\frac{\omega_2}{\Omega_0} + \tilde{\gamma} \frac{\beta\omega_1}{\alpha\Omega_0} \right) \cos \theta_0\end{aligned}\tag{2.36}$$

These expressions lead to the dispersion relation for the giant magnon solution on γ -deformed S^3 [33, 34]

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \cos^2 \theta_0}\tag{2.37}$$

In order to make a connection with the spin chain description we should identify $\cos \theta_0 = \sin \left(\frac{p}{2} - \pi\beta \right)$, where p is the momentum of the magnon excitation on the spin chain and $\beta = \tilde{\gamma}/\sqrt{\lambda}$. So the prediction for the relevant spin chain dispersion relation is

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2 \left(\frac{p}{2} - \pi\beta \right)}\tag{2.38}$$

this relation is invariant under $p \rightarrow p + 2\pi$ and $\beta \rightarrow \beta + 1$ as is required by the spin chain analysis [39, 40]. In [37] a detailed analysis of the infinite limit of the charges, as well as finite size corrections, is presented. It was argued that in the limit $\mathcal{J} = J/g \rightarrow \infty$ the dispersion relations does not feel the deformation since it shows up just as a shift by $\pi\gamma$. It is important however that the deformation produces a non-trivial twist in the boundary conditions for the isometry directions which is proportional to $\sim \gamma J$. The latter has non-trivial consequences, the analysis of which can be seen in [37].

Single spikes. The string profile with one single spike and a large winding number is realized when $\beta^2\kappa^2 = \alpha^2\omega_1^2$ and hence $A = -\frac{\omega_1\alpha^2}{\beta}$ [14]. It is natural to expect the existence

of a rigid string solution on S^3_γ which is the analogue of the single spike solution on S^3 found in [14]. The equations of motion are

$$\begin{aligned}\tilde{\phi}'_1 &= \frac{1}{\alpha^2 - \beta^2} \left(\beta\omega_1 - \frac{\alpha^2\omega_1}{\beta \sin^2 \theta} - \tilde{\gamma}\alpha\sqrt{\omega_1^2 - \Omega_1^2} \cos^2 \theta \right) \\ \tilde{\phi}'_1 &= \frac{1}{\alpha^2 - \beta^2} (\beta\omega_2 + \tilde{\gamma}\alpha\omega_1 \sin^2 \theta) \\ \theta' &= \frac{\alpha\Omega_1}{(\alpha^2 - \beta^2)} \frac{\cos \theta}{\sin \theta} \sqrt{\sin^2 \theta - \sin^2 \theta_1}\end{aligned}\tag{2.39}$$

where

$$\sin \theta_1 = \frac{\alpha\omega_1}{\beta\Omega_1} \quad \Omega_1 = \sqrt{\omega_1^2 - \left(\omega_2 + \tilde{\gamma} \frac{\alpha\omega_1}{\beta} \right)^2}\tag{2.40}$$

The two conserved angular momenta are

$$J_1 = 2T \frac{\omega_1}{\Omega_1} \cos \theta_1 \quad J_2 = -2T \frac{\sqrt{\omega_1^2 - \Omega_1^2}}{\Omega_1} \cos \theta_1\tag{2.41}$$

The relation between the conserved charges becomes

$$J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \cos^2 \theta_1}\tag{2.42}$$

This looks identical to the corresponding expression in the undeformed case, the dependence on the deformation parameter $\tilde{\gamma}$ is buried in the definition of $\cos \theta_1$. In analogy with the giant magnon solution we can identify $\cos \theta_1 = \sin \left(\frac{\beta}{2} - \pi\beta \right)$.

For the relation between E and $\Delta\phi_1$ we find:

$$E - T\Delta\phi_1 = \frac{\sqrt{\lambda}}{\pi} \left(\frac{\pi}{2} - \theta_1 \right) - \tilde{\gamma} \frac{\sqrt{\lambda}}{\pi} \frac{\sqrt{\omega_1^2 - \Omega_1^2}}{\Omega_1} \cos \theta_1\tag{2.43}$$

As should be expected that in the limit $\tilde{\gamma} \rightarrow 0$ this expression reduces to the one for the single spike solution on undeformed S^3 .

3 Giant magnon and single spike string solutions on the deformed $T^{1,1}$

In this section we present the classical solutions in a particular (consistent) subsector of the deformed conifold. First we will give a short set up of the beta-deformed conifold [30, 38]. Next we consider a solitonic ansatz for the giant magnon and single spike classical string solutions and find the explicit form of the solutions. At the end of this section we briefly comment on the motion of rigid folded strings in the deformed background.

Since the beta-deformed $T^{1,1}$ is known, we will quote here only its final form referring for instance to [30, 38]. The starting point of the deformation procedure is the metric of $AdS_5 \times T^{1,1}$

$$\frac{ds^2}{R^2} = ds^2_{AdS} + \frac{1}{6} \sum_{i=1}^2 (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \frac{1}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2.\tag{3.1}$$

Here we set the deformation parameter of the squashed sphere to $b = 2/3$, i.e. conifold. Note that there is no B-field.

According to the procedure, described in [30, 32], the deformed geometry can be obtained by applying T-duality and a shift followed by another T-duality. The whole procedure can be organized in a single transformation as in [30, 38] and the result is given by

$$\frac{ds^2}{R^2} = ds_{AdS}^2 + G \left[\frac{1}{6} \sum_{i=1}^2 (G^{-1} d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \frac{1}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 + \tilde{\gamma}^2 \frac{\sin^2 \theta_1 \sin^2 \theta_2}{324} d\psi^2 \right] \quad (3.2)$$

Due to the T-dualities a non-trivial B-field is generated

$$\frac{B}{R^2} = \tilde{\gamma} G \left[\left(\frac{\sin^2 \theta_1 \sin^2 \theta_2}{36} + \frac{\cos^2 \theta_1 \sin^2 \theta_2 + \cos^2 \theta_2 \sin^2 \theta_1}{54} \right) d\phi_1 \wedge d\phi_2 + \frac{\sin^2 \theta_1 \cos \theta_2}{54} d\phi_1 \wedge d\psi - \frac{\cos \theta_1 \sin^2 \theta_2}{54} d\phi_2 \wedge d\psi \right]. \quad (3.3)$$

The conformal factor in the metric and the B-field has the form¹

$$G^{-1} = 1 + \tilde{\gamma}^2 \left(\frac{\cos^2 \theta_1 \sin^2 \theta_2 + \cos^2 \theta_2 \sin^2 \theta_1}{54} + \frac{\sin^2 \theta_1 \sin^2 \theta_2}{36} \right). \quad (3.4)$$

3.1 Giant magnon and single spike string solutions

Let us start with some simplifications of the problem under consideration. The complete solution of the non-linear problem is a very complicated task so we will restrict ourselves to a certain subsector. As in the undeformed case one can check by direct inspection that the following ansatz is a consistent truncation of the complete background.

$$\theta_2 = \text{const.}, \quad \phi_2 = \text{const.} \quad (3.5)$$

To further simplify considerations we choose $\theta_2 = 0$. Next we choose the following *ansatz* for solitonic string configurations

$$\begin{aligned} t &= \kappa\tau, & \theta_2 &= 0, & \phi_2 &= \text{const.} \\ \Psi &= \omega_\psi \tau + \psi(y); & \Phi &= \omega_\phi \tau + \phi(y), & \theta &= \theta(y), \end{aligned} \quad (3.6)$$

where $y = c\sigma - d\tau$, Ψ is the U(1) fiber coordinate while $\Phi \equiv \phi_1$.

With this choice the metric becomes (we set $R^2 = 1$)

$$ds^2 = -dt^2 + \frac{1}{6} d\theta^2 + \frac{G}{6} \sin^2 \theta d\phi^2 + \frac{G}{9} (d\psi + \cos \theta d\phi)^2 \quad (3.7)$$

¹We skip here the rest of the field content since it will not be used in what follows.

and the B-field takes the form

$$B = G\tilde{\gamma}\frac{\sin^2\theta}{54}. \quad (3.8)$$

In (3.7) and (3.8) the factor G can be read off from (3.4)

$$G^{-1} = 1 + \tilde{\gamma}^2\frac{\sin^2\theta}{54}. \quad (3.9)$$

We are looking for solutions with the profile of arc or spike moving along the isometry directions and described by (3.6). The Lagrangian can be easily deduced from (3.7), (3.8) and takes the form

$$\begin{aligned} \mathcal{L} \sim & \dot{t}^2 + \frac{1}{6}(-\dot{\theta}^2 + \theta'^2) + \frac{G}{9}\left(1 + \frac{\sin^2\theta}{2}\right)(-\dot{\Phi}^2 + \Phi'^2) \\ & + \frac{G}{9}(-\dot{\Psi}^2 + \Psi'^2) + \frac{2G}{9}\cos\theta(-\dot{\Psi}\dot{\Phi} + \Psi'\Phi') + 2G\tilde{\gamma}\frac{\sin^2\theta}{54}(\dot{\Phi}\Psi' - \dot{\Psi}\Phi'). \end{aligned} \quad (3.10)$$

In terms of θ, ϕ and ψ it reads off

$$\begin{aligned} \mathcal{L} \sim & \dot{t}^2 + \frac{c^2 - d^2}{6}\theta'^2 + \frac{G}{9}\left(1 + \frac{\sin^2\theta}{2}\right)\left[-(\omega_\phi - d\phi')^2 + c^2\phi'^2\right] \\ & + 2\frac{G}{9}\cos\theta\left[-(\omega_\psi - d\psi')(\omega_\phi - d\phi') + c^2\psi'\phi'\right] + \frac{G}{9}\left[-(\omega_\psi - d\psi')^2 + c^2\psi'^2\right] \\ & + 2G\tilde{\gamma}\frac{\sin^2\theta}{54}c\left[(\omega_\phi - d\phi')\psi' - (\omega_\psi - d\psi')\phi'\right]. \end{aligned} \quad (3.11)$$

It is easy to vary the action and to obtain the equations of motion for ψ and ϕ . They can be integrated once providing expressions in terms of θ . Explicitly it goes as follows. For ψ we get

$$\partial_y \left\{ \frac{2G}{9} [d(\omega_\psi - d\psi') + c^2\psi'] + \frac{2G}{9} \cos\theta [d(\omega_\phi - d\phi') + c^2\phi'] + 2G\tilde{\gamma}\frac{\sin^2\theta}{54}c\omega_\phi \right\} = 0, \quad (3.12)$$

or

$$(c^2 - d^2)\psi' + d\omega_\psi + \cos\theta[(c^2 - d^2)\phi' + d\omega_\phi] + \tilde{\gamma}\frac{\sin^2\theta}{6}c\omega_\phi = \frac{9A_\psi}{2G}. \quad (3.13)$$

Analogously, for ϕ we find

$$\begin{aligned} \partial_y \left\{ \frac{2G}{9} \left(1 + \frac{\sin^2\theta}{2}\right) [d(\omega_\phi - d\phi') + c^2\phi'] \right. \\ \left. + \frac{2G}{9} \cos\theta [d(\omega_\psi - d\psi') + c^2\psi'] - 2G\tilde{\gamma}\frac{\sin^2\theta}{54}c\omega_\psi \right\} = 0, \end{aligned} \quad (3.14)$$

or

$$\left(1 + \frac{\sin^2\theta}{2}\right) [(c^2 - d^2)\phi' + d\omega_\phi] + \cos\theta[(c^2 - d^2)\psi' + d\omega_\psi] - \tilde{\gamma}\frac{\sin^2\theta}{6}c\omega_\psi = \frac{9A_\phi}{2G}. \quad (3.15)$$

It is easy to obtain expressions for ψ' and ϕ' separately, namely from (3.13) and (3.15) we get

$$(c^2 - d^2)\phi' + d\omega_\phi = \frac{3(A_\phi - A_\psi \cos \theta)}{G \sin^2 \theta} + \tilde{\gamma} \frac{c}{9} (\omega_\psi + \omega_\phi \cos \theta), \quad (3.16)$$

and

$$(c^2 - d^2)\psi' + \omega_\psi d = \frac{3A_\psi}{2G} + \frac{3(A_\psi - A_\phi \cos \theta)}{G \sin^2 \theta} - \frac{\tilde{\gamma} c \omega_\psi \cos \theta}{9} - \frac{\tilde{\gamma} c \omega_\phi}{9} \left(1 + \frac{\sin^2 \theta}{2}\right). \quad (3.17)$$

Virasoro constraints. One of the important issues are the Virasoro constraints. In the parameterization we work with, the Virasoro constraints have both, diagonal and off-diagonal components non-trivial. The diagonal part of the Virasoro constraints consists of $T_{\tau\tau} + T_{\sigma\sigma} = 0$

$$\frac{1}{6}(\dot{\theta}^2 + \theta'^2) + \frac{G}{3} \left(\frac{\sin^2 \theta}{2} + \frac{\cos^2 \theta}{3} \right) (\dot{\Phi}^2 + \Phi'^2) + \frac{G}{9}(\dot{\Psi}^2 + \Psi'^2) + \frac{2G}{9} \cos \theta (\dot{\Psi}\dot{\Phi} + \Psi'\Phi') = \kappa^2. \quad (3.18)$$

or

$$\begin{aligned} \frac{c^2 + d^2}{6} \theta'^2 + \frac{G}{9} \left(1 + \frac{\sin^2 \theta}{2}\right) [(\omega_\phi - d\phi')^2 + c^2 \phi'^2] \\ + 2\frac{G}{9} \cos \theta [(\omega_\psi - d\psi')(\omega_\phi - d\phi') + c^2 \psi' \phi'] + \frac{G}{9} [(\omega_\psi - d\psi')^2 + c^2 \psi'^2] = \kappa^2. \end{aligned} \quad (3.19)$$

For future use it is convenient to rewrite it in the form

$$\begin{aligned} \frac{1}{6} \theta'^2 + \frac{G}{9} \left(1 + \frac{\sin^2 \theta}{2}\right) \left[\phi'^2 + \frac{\omega_\phi^2 - 2d\omega_\phi \phi'}{c^2 + d^2} \right] \\ + 2\frac{G}{9} \cos \theta \left[\psi' \phi' + \frac{\omega_\psi \omega_\phi - d(\omega_\phi \psi' + \omega_\psi \phi')}{c^2 + d^2} \right] + \frac{G}{9} \left[\psi'^2 + \frac{\omega_\psi^2 - 2d\omega_\psi \psi'}{c^2 + d^2} \right] = \frac{\kappa^2}{c^2 + d^2}. \end{aligned} \quad (3.20)$$

The off-diagonal part is

$$\begin{aligned} -\frac{cd}{6} \theta'^2 + \frac{G}{9} \left(1 + \frac{\sin^2 \theta}{2}\right) (\omega_\phi - d\phi') c \phi' + \frac{G}{9} (\omega_\psi - d\psi') c \psi' \\ + \cos \theta \frac{G}{9} [(\omega_\psi - d\psi') \phi' + (\omega_\phi - d\phi') \psi'] c = 0. \end{aligned} \quad (3.21)$$

This can be rewritten as

$$\begin{aligned} \frac{1}{6} \theta'^2 + \frac{G}{9} \left(1 + \frac{\sin^2 \theta}{2}\right) \left(\phi'^2 - \frac{\omega_\phi \phi'}{d} \right) + \frac{G}{9} \left(\psi'^2 - \frac{\omega_\psi \psi'}{d} \right) \\ + \frac{G}{9} \cos \theta \left[2\psi' \phi' - \frac{\omega_\psi \phi' + \omega_\phi \psi'}{d} \right] = 0. \end{aligned} \quad (3.22)$$

Subtracting (3.22) from (3.21) we find

$$\begin{aligned} \frac{G}{9} \left(1 + \frac{\sin^2 \theta}{2}\right) \left[\omega_\phi^2 + \frac{(c^2 - d^2)\omega_\phi \phi'}{d} \right] + \frac{G}{9} \left[\omega_\psi^2 + \frac{(c^2 - d^2)\omega_\psi \psi'}{d} \right] \\ + \frac{G}{9} \cos \theta \left[2\omega_\psi \omega_\phi + \frac{(c^2 - d^2)(\omega_\psi \phi' + \omega_\phi \psi')}{d} \right] = \kappa^2. \end{aligned} \quad (3.23)$$

Substituting the explicit form of ϕ' and ψ' from (3.16) and (3.17) we find

$$\omega_\phi A_\phi + \omega_\psi A_\psi = 2\kappa^2 d. \quad (3.24)$$

This expression puts a string restriction on the parameters of the solutions.

Equation of motion for θ . The equation of motion for θ , obtained by varying the action, is more complicated since it contains the other dynamical variables. It is easier to use another way to obtain it - by making use of the Virasoro constraints. Here we will use the second (off-diagonal) Virasoro constraint to obtain the equation of motion for θ . The latter can be written in the form

$$\theta'^2 + \frac{2G}{3d} \left\{ (d\phi' - \omega_\phi) \left[\left(1 + \frac{\sin^2 \theta}{2}\right) \phi' + \cos \theta \psi' \right] + (d\psi' - \omega_\psi) \left[\psi' + \cos \theta \phi' \right] \right\} = 0. \quad (3.25)$$

From (3.15) we have

$$(c^2 - d^2) \left[\left(1 + \frac{\sin^2 \theta}{2}\right) \phi' + \cos \theta \psi' \right] = \frac{9A_\phi}{2G} + \tilde{\gamma} c \omega_\psi \frac{\sin^2 \theta}{6} - \left(1 + \frac{\sin^2 \theta}{2}\right) d\omega_\phi - \cos \theta d\omega_\psi. \quad (3.26)$$

From (3.13) we find

$$(c^2 - d^2) \left[\psi' + \cos \theta \phi' \right] = \frac{9A_\psi}{2G} - \tilde{\gamma} c \omega_\phi \frac{\sin^2 \theta}{6} - d\omega_\psi - d\omega_\phi \cos \theta. \quad (3.27)$$

On other hand

$$d\phi' - \omega_\phi = \left\{ \frac{3d(A_\phi - A_\psi \cos \theta)}{G \sin^2 \theta} + \tilde{\gamma} \frac{cd}{9} (\omega_\psi + \omega_\phi \cos \theta) - c^2 \omega_\phi \right\} / (c^2 - d^2). \quad (3.28)$$

and

$$d\psi' - \omega_\psi = \left\{ \frac{3dA_\psi}{2G} + \frac{3d(A_\psi - A_\phi \cos \theta)}{G \sin^2 \theta} - \frac{\tilde{\gamma} dc \omega_\psi \cos \theta}{9} - \frac{\tilde{\gamma} dc \omega_\phi}{9} \left(1 + \frac{\sin^2 \theta}{2}\right) - c^2 \omega_\psi \right\} / (c^2 - d^2). \quad (3.29)$$

Substituting into the equation (3.25) we find

$$\theta'^2 + \frac{2G}{3(c^2 - d^2)^2} \left\{ \dots + c^2 \omega_\psi (\omega_\psi + \omega_\phi \cos \theta) G^{-1} + c^2 \omega_\phi^2 \left(1 + \frac{\sin^2 \theta}{2}\right) G^{-1} + c^2 \omega_\psi \omega_\phi \cos \theta G^{-1} \right\} = 0, \quad (3.30)$$

where \dots are the terms proportional to G^{-1} and G^{-2} which come from direct multiplication by G^{-1} in (3.26), (3.29). The others are organized in G^{-1} as above.

In (3.30) the terms in the brackets proportional to G^{-2} are

$$\{\dots\}_{|\sim\frac{1}{G^2}} = \frac{27d}{2G^2 \sin^2 \theta} \left[A_\phi(A_\phi - A_\psi \cos \theta) + A_\psi(A_\psi - A_\phi \cos \theta) + A_\psi^2 \frac{\sin^2 \theta}{2} \right]. \quad (3.31)$$

The terms in the brackets in (3.30) proportional to $1/G$ get contributions from two sources. The contributions coming from ϕ

$$\{\dots\}_\phi = \frac{1}{2G \sin^2 \theta} \left[\sin^2 \theta \left[\tilde{\gamma} c d A_\phi (\omega_\psi + \omega_\phi \cos \theta) - 9c^2 \omega_\phi A_\phi \right. \right. \quad (3.32) \\ \left. \left. - (3d^2 \omega_f - \tilde{\gamma} d c \omega_\psi) (A_\phi - A_\psi \cos \theta) \right] - 6d^2 (A_\phi - A_\psi \cos \theta) (\omega_\phi + \omega_\psi \cos \theta) \right]$$

and terms proportional to $1/G$ coming from ψ

$$\{\dots\}_\psi = \frac{-1}{2G \sin^2 \theta} \left[\sin^2 \theta \left[\tilde{\gamma} c d A_\psi (\omega_\phi + \omega_\psi \cos \theta) + \tilde{\gamma} d c \omega_f A_\psi \sin^2 \theta + 9c^2 \omega_\psi A_\psi \right. \right. \\ \left. \left. + 3d^2 A_\psi (\omega_\psi + \omega_\phi \cos \theta) + \tilde{\gamma} d c \omega_\phi (A_\psi - A_\phi \cos \theta) \right] \right. \\ \left. + 6d^2 (A_\psi - A_\phi \cos \theta) (\omega_\psi + \omega_\phi \cos \theta) \right]. \quad (3.33)$$

To obtain the complete form of the equation we have to add all the terms (3.31), (3.33) and substitute into (3.30).

Let us write down the final form of the equation

$$\theta'^2 + \frac{1}{3(c^2 - d^2)^2 \sin^2 \theta} \left\{ \left[\frac{(2c\omega_\phi - \tilde{\gamma} A_\psi)^2}{4} \right] \sin^4 \theta \right. \\ + (2c\omega_\phi - \tilde{\gamma} A_\psi)(2c\omega_\psi + \tilde{\gamma} A_\phi) \cos \theta \sin^2 \theta \\ + \frac{1}{2} \left[(2c\omega_\psi + \tilde{\gamma} A_\phi)^2 + (2c\omega_\phi - \tilde{\gamma} A_\psi)^2 + 27A_\psi^2 \right. \\ \left. - \frac{18}{d} (c^2 + d^2) (A_\phi \omega_\phi + A_\psi \omega_\psi) \right] \sin^2 \theta \\ \left. - 54A_\phi A_\psi \cos \theta + 27(A_\psi^2 + A_\phi^2) \right\} = 0. \quad (3.34)$$

For future use we need an expression in terms of $\cos \theta$ only, i.e. the equation in terms of

$\cos \theta$ becomes

$$\begin{aligned} \theta'^2 + \frac{1}{3(c^2 - d^2)^2 \sin^2 \theta} \left\{ \left[\frac{(2c\omega_\phi - \tilde{\gamma}A_\psi)^2}{4} \right] \cos^4 \theta - (2c\omega_\phi - \tilde{\gamma}A_\psi)(2c\omega_\psi + \tilde{\gamma}A_\phi) \cos^3 \theta \right. \\ - \frac{1}{2} \left[(2c\omega_\psi + \tilde{\gamma}A_\phi)^2 + 2(2c\omega_\phi - \tilde{\gamma}A_\psi)^2 + 27A_\psi^2 \right. \\ \left. \left. - \frac{18}{d}(c^2 + d^2)(A_\phi\omega_\phi + A_\psi\omega_\psi) \right] \cos^2 \theta \right. \\ + [(2c\omega_\phi - \tilde{\gamma}A_\psi)(2c\omega_\psi + \tilde{\gamma}A_\phi) - 54A_\phi A_\psi] \cos \theta \\ \left. + \frac{1}{2} \left[(2c\omega_\psi + \tilde{\gamma}A_\phi)^2 + \frac{3}{2}(2c\omega_\phi - \tilde{\gamma}A_\psi)^2 \right. \right. \\ \left. \left. + 27(3A_\psi^2 + 2A_\phi^2) - \frac{18}{d}(c^2 + d^2)(A_\phi\omega_\phi + A_\psi\omega_\psi) \right] \right\} = 0. \end{aligned} \quad (3.35)$$

The turning point. As discussed in the Introduction, we are looking for solutions describing strings with certain profile, namely, arcs or spikes. Therefore, these must have turning points. Here we derive the relations following from the condition (3.34) to have a turning point at $\theta^* = \pi$.

One can see that the only singular terms at $\theta^* = \pi$ are those in the last line of (3.34). To cancel these singularities we must impose some conditions. The last line can be written as

$$\frac{27}{\sin^2 \theta} [(A_\phi - A_\psi \cos \theta)^2 + A_\psi^2 \sin^2 \theta]. \quad (3.36)$$

Then if

$$A_\psi = -A_\phi, \quad (3.37)$$

in the limit $\theta \rightarrow \pi$ the first term in the brackets vanishes as $\sim 0^4$ and the only finite contribution comes from the second term ($= 27A_\psi^2$). With (3.37) one can safely write down the turning point condition at $\theta^* = \pi$

$$\begin{aligned} \frac{1}{2}(2c\omega_\phi + \tilde{\gamma}A_\phi)^2 + \frac{1}{2}(2c\omega_\psi + \tilde{\gamma}A_\phi)^2 - (2c\omega_\phi + \tilde{\gamma}A_\phi)(2c\omega_\psi + \tilde{\gamma}A_\phi) + \frac{81}{2}A_\phi^2 \\ = \frac{18}{2d}(c^2 + d^2)A_\phi(\omega_\phi - \omega_\psi), \end{aligned} \quad (3.38)$$

or

$$81A_\phi^2 - 18\frac{c^2 + d^2}{d}A_\phi(\omega_\phi - \omega_\psi) + 4c^2(\omega_\phi - \omega_\psi)^2 = 0. \quad (3.39)$$

The last equation can be written as

$$\left(A_\phi - \frac{2}{9}d(\omega_\phi - \omega_\psi) \right) \left(A_\phi - \frac{2c^2}{9d}(\omega_\phi - \omega_\psi) \right) = 0. \quad (3.40)$$

From (3.40) we find two conditions

$$A_\phi = \begin{cases} \frac{2}{9}d(\omega_\phi - \omega_\psi) & \text{giant magnon} \\ \frac{2c^2}{9d}(\omega_\phi - \omega_\psi) & \text{single spike} \end{cases} \quad (3.41)$$

We remind that the last expression is accompanied by

$$A_\psi = -A_\phi.$$

The solution. Let us introduce, for convenience, the notations:

$$B_\psi = 2c\omega_\psi + \tilde{\gamma}A_\phi, \quad B_\phi = 2c\omega_\phi + \tilde{\gamma}A_\phi. \quad (3.42)$$

In terms of $\cos^2 \frac{\theta}{2} = u$ the equation (3.36) can be written in the following form:

$$4u'^2 + \frac{1}{3(c^2 - d^2)^2} \left\{ 4B_\phi^2 u^4 - 8B_\phi(B_\phi + B_\psi)u^3 \right. \\ \left. + 2 \left[B_\phi^2 - B_\psi^2 + 6B_\phi B_\psi - 27A_\phi^2 + \frac{18}{d}(c^2 + d^2)A_\phi(\omega_\phi - \omega_\psi) \right] u^2 \right. \\ \left. + 2 \left[(B_\psi - B_\phi)^2 + 81A_\phi^2 - \frac{18}{d}(c^2 + d^2)A_\phi(\omega_\phi - \omega_\psi) \right] u \right\} = 0. \quad (3.43)$$

The turning point condition (3.41) in these variables is

$$(B_\psi - B_\phi)^2 + 81A_\phi^2 - \frac{18}{d}(c^2 + d^2)A_\phi(\omega_\phi - \omega_\psi) = 0, \quad (3.44)$$

and therefore the last term in the equation (3.43) vanishes

$$u'^2 = \frac{1}{3(c^2 - d^2)^2} \left[-B_\phi^2 u^4 + 2B_\phi(B_\phi + B_\psi)u^3 - (B_\phi^2 + 2B_\phi B_\psi + 27A_\phi^2)u^2 \right]. \quad (3.45)$$

A simple analysis analogous to that in [54] shows that the equation (3.45) can be written as:

$$u'^2 = \frac{B_\phi^2}{3(c^2 - d^2)^2} u^2 (\alpha_> - u)(u + \alpha_-), \quad (3.46)$$

where

$$0 < \alpha_> = 1 + \frac{B_\psi}{B_\phi} \left(1 - \sqrt{1 - 27 \frac{A_\phi^2}{B_\psi^2}} \right) < 1, \\ \alpha_- = |\alpha_<| = -1 - \frac{B_\psi}{B_\phi} \left(1 + \sqrt{1 - 27 \frac{A_\phi^2}{B_\psi^2}} \right) > 0, \quad (3.47)$$

and $0 \leq u \leq \alpha_> < 1$.

The solution can be easily obtained and is given by:

$$u(y) = \left(\frac{2\alpha_>\alpha_-}{\alpha_> + \alpha_-} \right) \frac{1}{\cosh \left(|a| \sqrt{\alpha_>\alpha_-} y \right) - \frac{\alpha_> - \alpha_-}{\alpha_> + \alpha_-}}, \quad (3.48)$$

where

$$a^2 = \frac{B_\phi^2}{3(c^2 - d^2)^2}. \quad (3.49)$$

Having obtained the solutions with the desired profile, one can proceed with the dispersion relations.

4 Dispersion relations

In this section we derive the conserved charges and the corresponding dispersion relations. Having obtained the classical string solutions, it is easy to compute the conserved charges. Due to the specific regime we are working in, namely the very high energies corresponding to very long dual operators, some of them are finite but some are divergent. The two cases, giant magnons and single spikes differs in boundary conditions, i.e. the profile of the string propagating along the isometry directions.

4.1 Conserved charges

Let us start with computing the conserved quantities in the theory. By definition, the conserved momenta corresponding to the isometries are:

$$P_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \Psi)}, \quad P_\phi = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \Phi)}, \quad P_t = \frac{\partial \mathcal{L}}{\partial(\partial_\tau t)} \quad (4.1)$$

Their explicit form is given by

$$-\frac{2}{T}P_\psi = -\frac{2G}{9} \left[\partial_\tau \Psi + \cos \theta \partial_\tau \Phi + \tilde{\gamma} \frac{\sin^2 \theta}{6} \partial_\sigma \Phi \right], \quad (4.2)$$

$$-\frac{2}{T}P_\phi = -\frac{2G}{9} \left[\left(1 + \frac{\sin^2 \theta}{2} \right) \partial_\tau \Phi + \cos \theta \partial_\tau \Psi - \tilde{\gamma} \frac{\sin^2 \theta}{6} \partial_\sigma \Psi \right], \quad (4.3)$$

$$-\frac{1}{T}P_t = \partial_\tau t, \quad (4.4)$$

or, substituting for $\partial_{\sigma,\tau} \Psi$ and $\partial_{\sigma,\tau} \Phi$

$$-\frac{2}{T}P_\psi = \frac{2G}{9} \left[d\psi' - \omega_\psi + \cos \theta (d\phi' - \omega_\phi) - \tilde{\gamma} c \frac{\sin^2 \theta}{6} \phi' \right], \quad (4.5)$$

$$-\frac{2}{T}P_\phi = \frac{2G}{9} \left[\left(1 + \frac{\sin^2 \theta}{2} \right) (d\phi' - \omega_\phi) + \cos \theta (d\psi' - \omega_\psi) + \tilde{\gamma} c \frac{\sin^2 \theta}{6} \psi' \right], \quad (4.6)$$

$$-\frac{1}{T}P_t = \kappa. \quad (4.7)$$

The corresponding charges are defined by

$$J_\psi = \int_{-\infty}^{\infty} \frac{dy}{c} P_\psi = \frac{T}{9} \int_{-\infty}^{\infty} \frac{dy}{c} G \left[\omega_\psi - d\psi' + \cos \theta (\omega_\phi - d\phi') + \tilde{\gamma} c \frac{\sin^2 \theta}{6} \phi' \right], \quad (4.8)$$

$$J_\phi = \int_{-\infty}^{\infty} \frac{dy}{c} P_\phi = \frac{T}{9} \int_{-\infty}^{\infty} \frac{dy}{c} \left[\left(1 + \frac{\sin^2 \theta}{2} \right) (\omega_\phi - d\phi') + \cos \theta (\omega_\psi - d\psi') - \tilde{\gamma} c \frac{\sin^2 \theta}{6} \psi' \right], \quad (4.9)$$

$$E = - \int_{-\infty}^{\infty} \frac{dy}{c} P_t = T \int_{-\infty}^{\infty} \frac{dy}{c} \kappa. \quad (4.10)$$

In the rest of this subsection we will compute explicitly the above charges. Due to the specific limit of large quantum numbers some expressions are divergent and we will analyze them here. In order to obtain the dispersion relations we need to find certain finite combinations out of the divergent ones. Below we start this analysis.

Computation of P_ψ . To obtain the dispersion relations we need the explicit form of the conserved charges. Let us first find the explicit expression for P_ψ

$$\begin{aligned}
 -\frac{2}{T} P_\psi &= \frac{2G}{9(c^2 - d^2)} \left\{ \frac{3dA_\psi}{2G} + \frac{3d(A_\psi - A_\phi \cos \theta)}{G \sin^2 \theta} - \tilde{\gamma} \frac{dc}{9} (\omega_\phi + \omega_\psi \cos \theta) - \tilde{\gamma} \frac{dc\omega_\phi}{9} \cdot \frac{\sin^2 \theta}{2} \right. \\
 &\quad \left. - c^2 \omega_\psi + \frac{3d \cos \theta (A_\phi - A_\psi \cos \theta)}{G \sin^2 \theta} + \tilde{\gamma} \frac{cd}{9} (\omega_\phi \cos^2 \theta + \omega_\psi \cos \theta) \right. \\
 &\quad \left. - c^2 \omega_\phi \cos \theta - \tilde{\gamma} \frac{\sin^2 \theta}{6} c (c^2 - d^2) \phi' \right\} \quad (4.11) \\
 &= \frac{2G}{9(c^2 - d^2)} \left\{ \frac{9dA_\psi}{2G} - c^2 (\omega_\psi + \omega_\phi \cos \theta) - \tilde{\gamma} c \frac{\sin^2 \theta}{6} [(c^2 - d^2) \phi' + d\omega_\phi] \right\}.
 \end{aligned}$$

The expression in the square brackets can be replaced by the expression from (3.16). The result we find is

$$-\frac{2}{T} P_\psi = \frac{2G}{9(c^2 - d^2)} \left\{ \frac{9dA_\psi - \tilde{\gamma} c (A_\phi - A_\psi \cos \theta)}{2G} - c^2 (\omega_\psi + \omega_\phi \cos \theta) \left[1 + \tilde{\gamma}^2 \frac{\sin^2 \theta}{54} \right] \right\}. \quad (4.12)$$

One can observe that the expression in the square brackets is exactly G^{-1} , so the final explicit expression for the momentum P_ψ is

$$-\frac{2}{T} P_\psi = \frac{1}{9(c^2 - d^2)} \left[9dA_\psi - \tilde{\gamma} c (A_\phi - A_\psi \cos \theta) - 2c^2 (\omega_\psi + \omega_\phi \cos \theta) \right]. \quad (4.13)$$

Computation of P_ϕ . Here we derive the explicit form of P_ϕ . We start with (4.6)

$$\begin{aligned}
 -\frac{2}{T} P_\phi &= \frac{2G}{9} \left\{ \left(1 + \frac{\sin^2 \theta}{2} \right) (d\phi' - \omega_\phi) + \cos \theta (d\psi' - \omega_\psi) + \tilde{\gamma} c \frac{\sin^2 \theta}{6} \psi' \right\} \quad (4.14) \\
 &= \frac{2G}{9} \left\{ \left[\left(1 + \frac{\sin^2 \theta}{2} \right) \phi' + \cos \theta \psi' \right] - \left(1 + \frac{\sin^2 \theta}{2} \right) \omega_\phi - \cos \theta \omega_\psi + \tilde{\gamma} c \frac{\sin^2 \theta}{2} \psi' \right\}.
 \end{aligned}$$

The expression in the square brackets is given in (3.26) and its substitution into (4.15) gives

$$\begin{aligned}
 -\frac{2}{T} P_\phi &= \frac{2G}{9(c^2 - d^2)} \left\{ \frac{9dA_\phi}{2G} + \tilde{\gamma} \frac{\sin^2 \theta}{6} c [(c^2 - d^2) \psi' + d\omega_\psi] \right. \\
 &\quad \left. - c^2 \left(\left(1 + \frac{\sin^2 \theta}{2} \right) \omega_\phi + \cos \theta \omega_\psi \right) \right\} \quad (4.15)
 \end{aligned}$$

The expression in the square brackets is exactly that of (3.17). As a result, we find

$$-\frac{2}{T} P_\phi = \frac{2G}{9(c^2 - d^2)} \left\{ \frac{9dA_\phi + \tilde{\gamma}c \frac{\sin^2 \theta}{2} A_\psi + \tilde{\gamma}c(A_\psi - A_\phi \cos \theta)}{2G} - c^2 \left(\left(1 + \frac{\sin^2 \theta}{2}\right) \omega_\phi + \cos \theta \omega_\psi \right) \left[1 + \tilde{\gamma}^2 \frac{\sin^2 \theta}{54}\right] \right\}. \quad (4.16)$$

In the square brackets one can recognize the expression for G^{-1} , so the final expression takes the form

$$-\frac{2}{T} P_\phi = \frac{1}{9(c^2 - d^2)} \left[9dA_\phi - \tilde{\gamma}cA_\phi \cos \theta + \tilde{\gamma}c \left(1 + \frac{\sin^2 \theta}{2}\right) A_\psi - 2c^2 \left(\left(1 + \frac{\sin^2 \theta}{2}\right) \omega_\phi + \cos \theta \omega_\psi \right) \right]. \quad (4.17)$$

As in the undeformed case the momentum P_ψ is linear in $u(y) = \cos^2 \theta/2$, while the momentum P_ϕ is quadratic. It is expected that there is a part analogous to the dispersion relations in the undeformed case (with some $\tilde{\gamma}$ deformations), but we also expect to have additional terms. Only the explicit computations can answer the question - what is the meaning and importance of the deformation?

The angle amplitude. From (3.16) and (3.17) it is clear that integrating ψ' and ϕ' we get divergent result. As in the undeformed case, one can look for a finite expression combining the two angles ϕ' and ψ' .

To this end we define the following combination

$$\Delta\varphi = \int dy \frac{\phi' - \psi'}{2}. \quad (4.18)$$

Now we are going to find the explicit expressions for the integrand and analyze the eventual divergences.

We start with subtracting (3.16) and (3.17)

$$(c^2 - d^2)[\phi' - \psi'] = \frac{3(A_\phi - A_\psi)(1 + \cos \theta) - 3A_\psi \frac{\sin^2 \theta}{2}}{\sin^2 \theta} \left(1 + \frac{\tilde{\gamma}^2}{54} \sin^2 \theta\right) - d(\omega_\phi - \omega_\psi) + \tilde{\gamma} \frac{c}{9} (\omega_\phi + \omega_\psi)(1 + \cos \theta) + \tilde{\gamma} \frac{c\omega_\phi \sin^2 \theta}{9 \cdot 2}. \quad (4.19)$$

In order to obtain an uniform description, it is better to pass to a variable $u = \cos^2 \theta/2$ and use

$$1 + \cos \theta = 2u, \quad \sin^2 \theta = 4u(1 - u). \quad (4.20)$$

Thus, we find for the ‘‘angle deficit’’ the expression

$$(c^2 - d^2)[\phi' - \psi'] = \frac{6(A_\phi - A_\psi) - 6A_\psi(1 - u) - 4d(\omega_\phi - \omega_\psi)(1 - u)}{4(1 - u)} + \tilde{\gamma} \frac{2c}{9} (\omega_\phi - \omega_\psi)u + \frac{\tilde{\gamma}^2}{9} [(A_\phi - A_\psi)u - A_\psi(1 - u)u] + \tilde{\gamma} \frac{2c}{9} \omega_\phi(1 - u)u. \quad (4.21)$$

Using the condition for the turning point (3.37) we find

$$(c^2 - d^2)[\phi' - \psi'] = \frac{18A_\phi - 4d(\omega_\phi - \omega_\psi) + [4d(\omega_\phi - \omega_\psi) - 6A_\phi]u}{4(1-u)} + \tilde{\gamma} \frac{2c}{9}(\omega_\phi - \omega_\psi)u + \frac{\tilde{\gamma}^2}{9}[(A_\phi - A_\psi)u - A_\psi(1-u)u] + \tilde{\gamma} \frac{2c}{9}\omega_\phi(1-u)u. \quad (4.22)$$

The final expression, we will use in what follows, is

$$(c^2 - d^2)[\phi' - \psi'] = \frac{3A_\phi}{1-u} + \frac{3}{2}A_\phi - d(\omega_\phi - \omega_\psi) + \frac{\tilde{\gamma}}{9}(2B_\phi + B_\psi)u - \frac{\tilde{\gamma}}{9}B_\phi u^2. \quad (4.23)$$

Let us make a few remarks about the behavior of the angle deficit in the two cases we are going to analyze. It is easy to see that integrating $\sim 1/(1-u)$ we will obtain a divergent result. In the magnon case the divergent term $\sim 1/(1-u)$ *vanishes* due to (3.41) and the expression becomes finite. In the single spike case the angle deficit is still divergent as it should be (we consider configurations with large winding numbers). In what follows we will consider the two cases separately.

4.2 Dispersion relations for giant magnons

In this subsection we will derive the dispersion relations for the giant magnons in the deformed conifold. The giant magnons are characterized with certain conditions, namely for giant magnon string solutions we have

$$A_\phi = \frac{2}{9}d(\omega_\phi - \omega_\psi), \quad A_\psi = -A_\phi$$

which combined with the second Virasoro constraint (3.24) gives

$$\kappa = \frac{\omega_\phi - \omega_\psi}{3} \quad (4.24)$$

Next task is to compute the conserved quantities for this case.

Expression for P_ψ . The expression for P_ψ in terms of u is

$$-\frac{2}{T}P_\psi = -\frac{1}{9(c^2 - d^2)} \left[9d \left(A_\phi - \frac{2c^2}{9d}(\omega_\phi - \omega_\psi) \right) + (4c^2\omega_\phi + 2\tilde{\gamma}cA_\phi) u \right]. \quad (4.25)$$

For magnon choice of A_ϕ (3.41) we find

$$-\frac{2}{T}P_\psi = \frac{2}{3} \cdot \frac{\omega_\phi - \omega_\psi}{3} - \frac{1}{9(c^2 - d^2)} (4c^2\omega_\phi + 2\tilde{\gamma}cA_\phi) u. \quad (4.26)$$

We note that the first term is exactly κ , c.f. (4.24).

It is easy now to write the expression for P_ψ

$$-\frac{2}{T}P_\psi = \frac{2}{3} \cdot \frac{\omega_\phi - \omega_\psi}{3} - \frac{2c}{9(c^2 - d^2)} B_\phi u. \quad (4.27)$$

It is clear that integrating (4.27) to obtain the conserved charge we get divergent result. However, the combination

$$E + 3J_\psi = \frac{B_\phi T}{3(c^2 - d^2)} \int dy u, \quad (4.28)$$

is finite since the first term of P_ψ cancels against κ from P_t .

Expression for P_ϕ . The next ingredient we need for the dispersion relations is P_ϕ . We simply substitute the constants for the magnon case in the corresponding expression (4.17) and obtain

$$-\frac{2}{T} P_\phi = -\frac{2}{3} \cdot \frac{\omega_\phi - \omega_\psi}{3} - \frac{2c}{9(c^2 - d^2)} [(2c(\omega_\psi + \omega_\phi) + 2\tilde{\gamma}A_\phi)u - (2c\omega_\phi + \tilde{\gamma}A_\phi)u^2]. \quad (4.29)$$

Also, introducing B_ϕ and B_ψ as defined in (3.42) we get

$$-\frac{2}{T} P_\phi = -\frac{2}{3} \cdot \frac{\omega_\phi - \omega_\psi}{3} - \frac{2c}{9(c^2 - d^2)} [(B_\phi + B_\psi)u - B_\phi u^2]. \quad (4.30)$$

One can observe that the finite combination here is

$$E - 3J_\phi, \quad (4.31)$$

where the first term cancel κ from \mathcal{E} .

Expression for $\Delta\varphi$. As we already mentioned in the last subsection, the angle deficit $\Delta\varphi$ defined by (4.18) is finite. Let us derive the explicit expression for $\Delta\varphi$.

The integrand in (4.18) in the magnon case can be derived by just substituting the corresponding values for the constants in (4.23)

$$\begin{aligned} (c^2 - d^2)[\phi' - \psi'] &= \frac{2d}{3}(\omega_\phi - \omega_\psi) \frac{u}{1-u} + \left(3\tilde{\gamma}^2 A_\phi + 2\tilde{\gamma}c(2\omega_\phi + \omega_\psi)\right) \frac{u}{9} \\ &\quad - \left(\tilde{\gamma}^2 A_\phi + 2\tilde{\gamma}c\omega_\phi\right) \frac{u^2}{9} \\ &= \frac{2}{3}d(\omega_\phi - \omega_\psi) \frac{u}{1-u} + \frac{\tilde{\gamma}}{9}(2B_\phi + B_\psi)u - \frac{\tilde{\gamma}}{9}B_\phi u^2. \end{aligned} \quad (4.32)$$

Since the combinations

$$\frac{E}{T} + 3\frac{J_\psi}{T} = \frac{1}{3(c^2 - d^2)} B_\phi \int_{-\infty}^{\infty} u dy \quad (4.33)$$

$$\frac{E}{T} - 3\frac{J_\phi}{T} = \frac{1}{3(c^2 - d^2)} B_\phi \int_{-\infty}^{\infty} u^2 dy - \frac{1}{3(c^2 - d^2)} (B_\phi + B_\psi) \int_{-\infty}^{\infty} u dy, \quad (4.34)$$

are finite, one can write the integrand as:

$$(c^2 - d^2) [\phi' - \psi'] = \frac{2}{3} d(\omega_\phi - \omega_\psi) \frac{u}{1-u} + \tilde{\gamma} \frac{(c^2 - d^2)}{3c} \left[\frac{1}{T} (-P_t) + 3 \frac{P_\psi}{T} \right] - \tilde{\gamma} \frac{(c^2 - d^2)}{3c} \left[\frac{1}{T} (-P_t) - 3 \frac{P_\phi}{T} \right]. \quad (4.35)$$

The angle amplitude then takes the following final form

$$\Delta\varphi = \int_{-\infty}^{\infty} dy \frac{\phi' - \psi'}{2} = \frac{d(\omega_\phi - \omega_\psi)}{3(c^2 - d^2)} \int_{-\infty}^{\infty} dy \frac{u}{1-u} + \frac{\tilde{\gamma}}{6} \left[\frac{E}{T} + 3 \frac{P_\psi}{T} \right] - \frac{\tilde{\gamma}}{6} \left[\frac{E}{T} - 3 \frac{P_\phi}{T} \right]. \quad (4.36)$$

The dispersion relations. Let us define the charge densities as $E/T = \mathcal{E}$, $J_\psi/T = \mathcal{J}_\psi$, $J_\phi/T = \mathcal{J}_\phi$. Then the finite combination of charges take the form:

$$\mathcal{E} + 3\mathcal{J}_\psi = \frac{\sqrt{3}}{3} a I_1, \quad (4.37)$$

$$\mathcal{E} - 3\mathcal{J}_\phi = \frac{\sqrt{3}}{3} a I_2 - \frac{\sqrt{3}}{3} a \frac{(\alpha_> - \alpha_-)}{2} I_1, \quad (4.38)$$

$$\Delta\varphi = \sqrt{(1 + \alpha_-)(1 - \alpha_>)} \frac{a}{2} I_3 + \frac{\tilde{\gamma}}{2} (\mathcal{J}_\psi + \mathcal{J}_\phi), \quad (4.39)$$

where the integrals I_i are given in the appendix and α_- and $\alpha_>$ are defined in (3.47). Using the explicit form of the integrals I_i (A.2), (A.4) we find:

$$\mathcal{E} + 3\mathcal{J}_\psi = \frac{2\sqrt{3}}{3} \arccos \left(\frac{\alpha_- - \alpha_>}{\alpha_- + \alpha_>} \right), \quad (4.40)$$

$$\mathcal{E} - 3\mathcal{J}_\phi = \frac{2\sqrt{3}}{3} \sqrt{\alpha_> \alpha_-}, \quad (4.41)$$

$$\Delta\varphi = \arccos \left(\frac{\alpha_- - \alpha_>}{\alpha_- + \alpha_>} - \frac{2\alpha_> \alpha_-}{\alpha_- + \alpha_>} \right) + \frac{\tilde{\gamma}}{2} (\mathcal{J}_\psi + \mathcal{J}_\phi). \quad (4.42)$$

From here we find that the constants $\alpha_>$ and α_- are related to the charge densities as follows:

$$\sqrt{\alpha_> \alpha_-} = \frac{3}{2\sqrt{3}} (\mathcal{E} - 3\mathcal{J}_\phi), \quad (4.43)$$

$$\frac{\alpha_- - \alpha_>}{\alpha_- + \alpha_>} = \cos \left[\frac{3}{2\sqrt{3}} (\mathcal{E} + 3\mathcal{J}_\psi) \right]. \quad (4.44)$$

Simple algebraic calculations² lead to

$$\cos \left[\frac{3}{2\sqrt{3}} (\mathcal{E} + 3\mathcal{J}_\psi) \right] - \frac{3}{2\sqrt{3}} (\mathcal{E} - 3\mathcal{J}_\phi) \sin \left[\frac{3}{2\sqrt{3}} (\mathcal{E} + 3\mathcal{J}_\psi) \right] = \cos \left(\Delta\varphi - \frac{\tilde{\gamma}}{2} (\mathcal{J}_\phi + \mathcal{J}_\psi) \right). \quad (4.45)$$

²We use for instance that $(\alpha_- - \alpha_>)/(\alpha_> + \alpha_-) - \sqrt{\alpha_> \alpha_-} \sqrt{1 - ((\alpha_- - \alpha_>)/(\alpha_> + \alpha_-))^2} = \cos(\Delta\varphi - \tilde{\gamma}/2(\mathcal{J}_\phi + \mathcal{J}_\psi))$.

The final form of the dispersion relations in the magnon case is

$$\frac{\sqrt{3}}{2}(\mathcal{E} - 3\mathcal{J}_\phi) = \frac{\cos\left[\frac{\sqrt{3}}{2}(\mathcal{E} + 3\mathcal{J}_\psi)\right] - \cos(\Delta\varphi - \tilde{\gamma}/2(\mathcal{J}_\phi + \mathcal{J}_\psi))}{\sin\left[\frac{\sqrt{3}}{2}(\mathcal{E} + 3\mathcal{J}_\psi)\right]}. \quad (4.46)$$

To close this subsection, let us make some short comments. First of all, the transcendental character of the dispersion relation persists in the deformed background. The deformation parameter enters the expression by shifting the angle amplitude by term proportional to γ times the total spin.³ The BMN and basic giant magnon analysis considered in [54] can be easily repeated with the same conclusions (up to the *gamma* shift). Note that each conserved charge depends on the γ parameter, but this dependence is hidden in the dispersion relations.

4.3 Dispersion relations for single spike strings

To obtain the dispersion relation for the single spike strings we have to compute the conserved quantities with the parameters describing strings with large winding number. This requirement leads to the relations (3.41) between the parameters.

Let us start with the condition for single spike string solutions

$$A_\phi = \frac{2c^2}{9d}(\omega_\phi - \omega_\psi),$$

which combined with the second Virasoro constraint (3.24) gives

$$\kappa = \frac{c(\omega_\phi - \omega_\psi)}{3d}. \quad (4.47)$$

Next, we have to compute all the conserved charges with these adjustments of the parameters.

Expressions for P_ψ and P_ϕ . We start with the expression for P_ψ . The simple substitution of the above values for the parameters into (4.13) gives

$$-\frac{2}{T}P_\psi = -\frac{1}{9(c^2 - d^2)}\left[9dA_\phi + 2c^2(\omega_\psi - \omega_\phi) + 2cB_\phi u\right]. \quad (4.48)$$

Analogously, the expression for P_ϕ derived from (4.17) has the form

$$-\frac{2}{T}P_\phi = \frac{1}{9(c^2 - d^2)}\left[9dA_\phi + 2c^2(\omega_\psi - \omega_\phi) - 2c(B_\phi + B_\psi)u + 2cB_\phi u^2\right]. \quad (4.49)$$

³In [37] another regime where the γ deformation scales to zero is realized

Expression for $\Delta\varphi$. The general expression for the angle amplitude

$$(c^2 - d^2)[\phi' - \psi'] = \frac{3A_\phi}{1-u} + \frac{3}{2}A_\phi - d(\omega_\phi - \omega_\psi) + \frac{\tilde{\gamma}}{9}(2B_\phi + B_\psi)u - \frac{\tilde{\gamma}}{9}B_\phi u^2 \quad (4.50)$$

in the case of a single spike profile the solution takes the form

$$(c^2 - d^2)[\phi' - \psi'] = 3A_\phi \frac{u}{1-u} + 3\frac{(c^2 - d^2)}{c}\kappa + \frac{\tilde{\gamma}}{9}(2B_\phi + B_\psi)u - \frac{\tilde{\gamma}}{9}B_\phi u^2. \quad (4.51)$$

The dispersion relations. The conserved quantities are

$$\frac{1}{T}P_\psi = \frac{1}{9}\frac{c}{(c^2 - d^2)}B_\phi u \quad (4.52)$$

$$\frac{1}{T}P_\phi = \frac{1}{9}\frac{c}{(c^2 - d^2)}(B_\phi + B_\psi)u - \frac{1}{9}\frac{c}{(c^2 - d^2)}B_\phi u^2. \quad (4.53)$$

One can see that the charges obtained by integrating the spins in this case are finite:

$$\frac{1}{T}J_\psi = \frac{1}{9(c^2 - d^2)}B_\phi \int_{-\infty}^{\infty} dyu \quad (4.54)$$

$$\frac{1}{T}J_\phi = \frac{1}{9(c^2 - d^2)}(B_\phi + B_\psi) \int_{-\infty}^{\infty} dyu - \frac{1}{9(c^2 - d^2)}B_\phi \int_{-\infty}^{\infty} dyu^2. \quad (4.55)$$

Therefore, the total momentum density

$$\frac{1}{T}P_\psi + \frac{1}{T}P_\phi = \frac{1}{9}\frac{c}{(c^2 - d^2)}(2B_\phi + B_\psi)u - \frac{1}{9}\frac{c}{(c^2 - d^2)}B_\phi u^2, \quad (4.56)$$

also defines a finite charge.

As in the undeformed case, the situation with the angle amplitude is more tricky. From (4.51) and (4.48), (4.49) one finds

$$(c^2 - d^2)[\phi' - \psi'] = 3A_\phi \frac{u}{1-u} + 3\frac{(c^2 - d^2)}{c}\left(-\frac{P_t}{T}\right) + \tilde{\gamma}\frac{(c^2 - d^2)}{c}\left(\frac{1}{T}P_\psi + \frac{1}{T}P_\phi\right). \quad (4.57)$$

The angle amplitude then is given by

$$\Delta\varphi = \int_{-\infty}^{\infty} dy \frac{\phi' - \psi'}{2} = \frac{3A_\phi}{2(c^2 - d^2)} \int_{-\infty}^{\infty} dy \frac{u}{1-u} + \frac{3}{2}\left(\frac{E}{T}\right) + \frac{\tilde{\gamma}}{2}\left(\frac{J_\psi}{T} + \frac{J_\phi}{T}\right). \quad (4.58)$$

We showed explicitly that the angle amplitude itself is divergent (as is the energy), but there exists a finite combination that can be used to find the dispersion relations. From (4.58) we see that the following combination is a finite:

$$\Delta\delta \equiv \Delta\varphi - \frac{3}{2}\left(\frac{E}{T}\right) - \frac{\tilde{\gamma}}{2}\left(\frac{J_\psi}{T} + \frac{J_\phi}{T}\right) = \frac{3A_\phi}{2(c^2 - d^2)} \int_{-\infty}^{\infty} dy \frac{u}{1-u}. \quad (4.59)$$

It is useful to introduce again the charge densities $E/T = \mathcal{E}$, $J_\psi/T = \mathcal{J}_\psi$, $J_\phi/T = \mathcal{J}_\phi$, which are found to be

$$\mathcal{J}_\psi = \frac{\sqrt{3}}{9}aI_1, \tag{4.60}$$

$$\mathcal{J}_\phi = -\frac{\sqrt{3}}{9}aI_2 + \frac{\sqrt{3}}{9}a\frac{(\alpha_> - \alpha_-)}{2}I_1, \tag{4.61}$$

$$\Delta\delta \equiv \Delta\varphi - \frac{3}{2}\mathcal{E} - \frac{\tilde{\gamma}}{2}(\mathcal{J}_\psi + \mathcal{J}_\phi) = \sqrt{(1 + \alpha_-)(1 - \alpha_>)}\frac{a}{2}I_3. \tag{4.62}$$

The substitution of the explicit form of the integrals I_i gives

$$\mathcal{J}_\psi = \frac{2\sqrt{3}}{9} \arccos\left(\frac{\alpha_- - \alpha_>}{\alpha_- + \alpha_>}\right), \tag{4.63}$$

$$\mathcal{J}_\phi = -\frac{2\sqrt{3}}{9}\sqrt{\alpha_>\alpha_-}, \tag{4.64}$$

$$\Delta\delta = \arccos\left(\frac{\alpha_- - \alpha_>}{\alpha_- + \alpha_>} - \frac{2\alpha_>\alpha_-}{\alpha_- + \alpha_>}\right). \tag{4.65}$$

One can again express the constants $\alpha_>$ and α_- in terms of the charge densities

$$-\sqrt{\alpha_>\alpha_-} = \frac{3\sqrt{3}}{2}\mathcal{J}_\phi, \tag{4.66}$$

$$\frac{\alpha_- - \alpha_>}{\alpha_- + \alpha_>} = \cos\left(\frac{3\sqrt{3}}{2}\mathcal{J}_\psi\right). \tag{4.67}$$

Using that

$$\frac{\alpha_- - \alpha_>}{\alpha_- + \alpha_>} - \sqrt{\alpha_>\alpha_-}\sqrt{1 - \left(\frac{\alpha_- - \alpha_>}{\alpha_- + \alpha_>}\right)^2} = \cos \Delta\delta. \tag{4.68}$$

we find

$$\cos\left(\frac{3\sqrt{3}}{2}\mathcal{J}_\psi\right) + \frac{3\sqrt{3}}{2}\mathcal{J}_\phi \sin\left(\frac{3\sqrt{3}}{2}\mathcal{J}_\psi\right) = \cos \Delta\delta. \tag{4.69}$$

The final form of the dispersion relations is

$$-\frac{3\sqrt{3}}{2}\mathcal{J}_\phi = \frac{\cos\left(\frac{3\sqrt{3}}{2}\mathcal{J}_\psi\right) - \cos \Delta\delta}{\sin\left(\frac{3\sqrt{3}}{2}\mathcal{J}_\psi\right)}. \tag{4.70}$$

The transcendental character of the dispersion relations persists as expected. The non-trivial shift of the angle amplitude is of the same form as in the case of the giant magnons and therefore has an universal form.

5 Conclusions

In this paper we have studied the problem of existence of a certain class solitonic solutions of strings in the beta-deformed $T^{1,1}$ which is the base of the conifold. The latter is an important example of a string dual of gauge theory with less than $\mathcal{N} = 4$ supersymmetry and has many interesting applications.

To set up the notations and make the paper more self contained, first we give a short review of the magnon and single spike solutions in the undeformed case as well as the magnon and spiky solutions in γ -deformed sphere S_γ^3 . In the following sections we present our original results, which can be summarized as follows. In section 3 we derive and analyze the classical string solitons of the magnon and single spike strings type for a subsector of the γ -deformed conifold. In the next section we obtain the dispersion relations for the classical solution found in the previous section. The results show that the dispersion relations are of the same transcendental type as the ones in the undeformed case [54]. The explicit dependence on γ shows up as a shift of the (generalized) angular amplitude - a behavior familiar from the studies of most supersymmetric case of strings in deformed $AdS_5 \times S^5$ background [33–36].

There are two essential differences from the known result of giant magnon and single spike strings in $AdS_5 \times S^5$. The first one is that the dispersion relations relate the conserved charges in a transcendental way in both cases - undeformed and γ -deformed conifold. This indicates that if there are integrable structures on the conifold they will be much more complicated than the known from the most supersymmetric case. The second difference is that, although the dispersion relations explicitly “feels” the γ -deformation through a shift in the angular extend of the magnon profile or winding number, in our case it has a qualitatively new feature. While in the case of sphere the shift was just by $\gamma\pi^4$ and one can turn on the regime of large charges in which classically the γ term scales to zero [37], here this shift is quite different. As shown in (4.46) and (4.70), it is proportional to the total momentum, which in our regime of validity is of the order of energy, i.e. very large. It seems that in our case this contribution cannot be made vanishing and it has exactly the form of the non-trivial twist of the boundary conditions. In any case, it is interesting that the shift involves also the conserved charges, a feature which deserves to be thoroughly analyzed.

An important issue to pursue is the search for integrable structures. This is an important open question which, if positive, would have important contribution to the understanding of the string/gauge theory duality. Another direction is to extend the analysis in this paper to the dynamics on the full $T^{1,1}$ manifolds [57] which is less ambitious but also very important.

Finally, it is known that there exists an one-parameter family of $AdS_5 \times X^5$ solutions which interpolates between the Klebanov-Witten background and the Pilch-Warner background [58]. It would be interesting to look for solitonic solutions in Pilch-Warner back-

⁴Note that actually we also used the identification $\tilde{\gamma} = \sqrt{\alpha}\gamma$.

ground, more over that not too much is known about quasiclassical strings in this background [60].

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A Useful formulae

Some integrals. Let us write down some useful integrals used in the calculations. Before that we stress on the following point. In accordance with the equations of motion we are dealing with solitary wave solution $u(y) = \cos^2 \theta/2$. The turning point $\theta(y = 0) = \theta_0$ corresponds to $u(y = 0) = \alpha_> = \cos^2 \frac{\theta_0}{2}$ while the turning point $\theta(y = \infty) = \pi$ corresponds to $u(y = \infty) = 0$. Note that when y increases from 0 to ∞ , $\theta(y)$ increase from θ_0 to π ($0 < \theta_0 \leq \theta(y) \leq \pi$), and hence the function $u(y) = \cos^2 \frac{\theta(y)}{2}$ decrease from $\alpha_>$ to 0 ($1 > \alpha \geq u(y) \geq 0$).

In computing conserved charges we need three integrals. First integral entering the calculations of the dispersion relations is

$$I_1 = \int_{-\infty}^{\infty} u dy. \tag{A.1}$$

To calculate the integral we use (3.46) and obtain

$$I_1 = \frac{2}{|a|} \arctan \left(\frac{2\sqrt{\alpha_>\alpha_-}}{\alpha_- - \alpha_>} \right) = \frac{2}{|a|} \arccos \left(\frac{\alpha_- - \alpha_>}{\alpha_- + \alpha_>} \right) = \frac{4}{|a|} \arccos \sqrt{\frac{\alpha_-}{\alpha_> + \alpha_-}} \tag{A.2}$$

Another integral appearing in our calculations is

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} u^2 dy = \frac{2}{|a|} \frac{(\alpha_> - \alpha_-)}{2} \arctan \left(\frac{2\sqrt{\alpha_>\alpha_-}}{\alpha_- - \alpha_>} \right) + \frac{2}{|a|} \sqrt{\alpha_>\alpha_-} = \\ &= \frac{2}{|a|} \frac{(\alpha_> - \alpha_-)}{2} \arccos \left(\frac{\alpha_- - \alpha_>}{\alpha_- + \alpha_>} \right) + \frac{2}{|a|} \sqrt{\alpha_>\alpha_-} \end{aligned} \tag{A.3}$$

We also used

$$\begin{aligned}
 I_3 &= \int_{-\infty}^{\infty} \frac{u}{1-u} dy \\
 &= \frac{2}{|a|} \frac{1}{\sqrt{(1+\alpha_-)(1-\alpha_+)}} \arctan \left(\frac{2\sqrt{\alpha_+\alpha_-(1+\alpha_-)(1-\alpha_+)}}{\alpha_- - \alpha_+ - 2\alpha_+\alpha_-} \right) \\
 &= \frac{2}{|a|} \frac{1}{\sqrt{(1+\alpha_-)(1-\alpha_+)}} \arccos \left(\frac{\alpha_- - \alpha_+}{\alpha_- + \alpha_+} - \frac{2\alpha_+\alpha_-}{\alpha_- + \alpha_+} \right). \tag{A.4}
 \end{aligned}$$

Some relations. There is a relation between I_1 and I_2 , which has the form

$$I_2 = \frac{\alpha_+ - \alpha_-}{2} I_1 - \frac{1}{a} \sqrt{\alpha_+\alpha_-}. \tag{A.5}$$

Also, we have the relations

$$a = \frac{B_\phi}{\sqrt{3}(c^2 - d^2)}, \quad \frac{(\alpha_+ - \alpha_-)}{2} = 1 + \frac{B_\psi}{B_\phi}, \quad \sqrt{(1+\alpha_-)(1-\alpha_+)} = \frac{3\sqrt{3}A_\phi}{B_\phi}. \tag{A.6}$$

References

- [1] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)] [[SPIRES](#)];
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [[hep-th/9802109](#)] [[SPIRES](#)];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[SPIRES](#)].
- [2] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D 69** (2004) 046002 [[hep-th/0305116](#)] [[SPIRES](#)].
- [3] D.E. Berenstein, J.M. Maldacena and H.S. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *JHEP* **04** (2002) 013 [[hep-th/0202021](#)] [[SPIRES](#)].
- [4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [[hep-th/0204051](#)] [[SPIRES](#)].
- [5] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [[hep-th/0212208](#)] [[SPIRES](#)].
- [6] M. Staudacher, *The factorized S -matrix of CFT/AdS*, *JHEP* **05** (2005) 054 [[hep-th/0412188](#)] [[SPIRES](#)].
- [7] N. Beisert, *The $SU(2|2)$ dynamic S -matrix*, *Adv. Theor. Math. Phys.* **12** (2008) 945 [[hep-th/0511082](#)] [[SPIRES](#)].
- [8] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [[hep-th/0406256](#)] [[SPIRES](#)].
- [9] G. Arutyunov and S. Frolov, *On $AdS_5 \times S^5$ string S -matrix*, *Phys. Lett.* **B 639** (2006) 378 [[hep-th/0604043](#)] [[SPIRES](#)].

- [10] G. Arutyunov, S. Frolov and M. Zamaklar, *The Zamolodchikov-Faddeev algebra for $AdS_5 \times S^5$ superstring*, *JHEP* **04** (2007) 002 [[hep-th/0612229](#)] [[SPIRES](#)].
- [11] R.A. Janik, *The $AdS_5 \times S^5$ superstring worldsheet S -matrix and crossing symmetry*, *Phys. Rev. D* **73** (2006) 086006 [[hep-th/0603038](#)] [[SPIRES](#)].
- [12] D.M. Hofman and J.M. Maldacena, *Giant magnons*, *J. Phys. A* **39** (2006) 13095 [[hep-th/0604135](#)] [[SPIRES](#)].
- [13] M. Kruczenski, *Spiky strings and single trace operators in gauge theories*, *JHEP* **08** (2005) 014 [[hep-th/0410226](#)] [[SPIRES](#)].
- [14] R. Ishizeki and M. Kruczenski, *Single spike solutions for strings on S^2 and S^3* , *Phys. Rev. D* **76** (2007) 126006 [[arXiv:0705.2429](#)] [[SPIRES](#)].
- [15] H.-Y. Chen, N. Dorey and K. Okamura, *Dyonic giant magnons*, *JHEP* **09** (2006) 024 [[hep-th/0605155](#)] [[SPIRES](#)].
- [16] M. Blau, J.M. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, *Class. Quant. Grav.* **19** (2002) L87 [[hep-th/0201081](#)] [[SPIRES](#)].
- [17] J.C. Plefka, *Lectures on the plane-wave string/gauge theory duality*, *Fortsch. Phys.* **52** (2004) 264 [[hep-th/0307101](#)] [[SPIRES](#)].
- [18] D. Sadri and M.M. Sheikh-Jabbari, *The plane-wave/super Yang-Mills duality*, *Rev. Mod. Phys.* **76** (2004) 853 [[hep-th/0310119](#)] [[SPIRES](#)].
- [19] N. Dorey, *Magnon bound states and the AdS/CFT correspondence*, *J. Phys. A* **39** (2006) 13119 [[hep-th/0604175](#)] [[SPIRES](#)].
- [20] G. Arutyunov, S. Frolov and M. Zamaklar, *Finite-size effects from giant magnons*, *Nucl. Phys. B* **778** (2007) 1 [[hep-th/0606126](#)] [[SPIRES](#)].
- [21] J.A. Minahan, A. Tirziu and A.A. Tseytlin, *Infinite spin limit of semiclassical string states*, *JHEP* **08** (2006) 049 [[hep-th/0606145](#)] [[SPIRES](#)].
- [22] M. Kruczenski, J. Russo and A.A. Tseytlin, *Spiky strings and giant magnons on S^5* , *JHEP* **10** (2006) 002 [[hep-th/0607044](#)] [[SPIRES](#)].
- [23] H. Dimov and R.C. Rashkov, *On the anatomy of multi-spin magnon and single spike string solutions*, *Nucl. Phys. B* **799** (2008) 255 [[arXiv:0709.4231](#)] [[SPIRES](#)].
- [24] K. Okamura and R. Suzuki, *A perspective on classical strings from complex sine-Gordon solitons*, *Phys. Rev. D* **75** (2007) 046001 [[hep-th/0609026](#)] [[SPIRES](#)].
- [25] M. Grigoriev and A.A. Tseytlin, *Pohlmeyer reduction of $AdS_5 \times S^5$ superstring σ -model*, *Nucl. Phys. B* **800** (2008) 450 [[arXiv:0711.0155](#)] [[SPIRES](#)].
- [26] N. Beisert, R. Hernandez and E. Lopez, *A crossing-symmetric phase for $AdS_5 \times S^5$ strings*, *JHEP* **11** (2006) 070 [[hep-th/0609044](#)] [[SPIRES](#)].
- [27] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* (2007) P01021 [[hep-th/0610251](#)] [[SPIRES](#)].
- [28] J.M. Maldacena and I. Swanson, *Connecting giant magnons to the pp-wave: an interpolating limit of $AdS_5 \times S^5$* , *Phys. Rev. D* **76** (2007) 026002 [[hep-th/0612079](#)] [[SPIRES](#)].
- [29] S. Benvenuti and E. Tonni, *Near-flat space limit and Einstein manifolds*, *JHEP* **02** (2008) 022 [[arXiv:0707.1676](#)] [[SPIRES](#)];
M. Kreuzer, C. Mayrhofer and R.C. Rashkov, *A note on the near flat limit for strings in the Maldacena-Núñez background*, *Phys. Rev. D* **77** (2008) 066016 [[arXiv:0711.3600](#)] [[SPIRES](#)];
M. Kreuzer, R.C. Rashkov and M. Schimpf, *Near flat space limit of strings on $AdS_4 \times CP^3$* ,

- Eur. Phys. J. C* **60** (2009) 471 [[arXiv:0810.2008](#)] [[SPIRES](#)].
- [30] O. Lunin and J.M. Maldacena, *Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [[hep-th/0502086](#)] [[SPIRES](#)].
- [31] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional $N = 1$ supersymmetric gauge theory*, *Nucl. Phys. B* **447** (1995) 95 [[hep-th/9503121](#)] [[SPIRES](#)].
- [32] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069 [[hep-th/0503201](#)] [[SPIRES](#)].
- [33] C.-S. Chu, G. Georgiou and V.V. Khoze, *Magnons, classical strings and β -deformations*, *JHEP* **11** (2006) 093 [[hep-th/0606220](#)] [[SPIRES](#)].
- [34] N.P. Bobev and R.C. Rashkov, *Multispin giant magnons*, *Phys. Rev. D* **74** (2006) 046011 [[hep-th/0607018](#)] [[SPIRES](#)].
- [35] N.P. Bobev and R.C. Rashkov, *Spiky strings, giant magnons and β -deformations*, *Phys. Rev. D* **76** (2007) 046008 [[arXiv:0706.0442](#)] [[SPIRES](#)].
- [36] N.P. Bobev, H. Dimov and R.C. Rashkov, *Semiclassical strings in Lunin-Maldacena background*, [hep-th/0506063](#) [[SPIRES](#)].
- [37] D.V. Bykov and S. Frolov, *Giant magnons in TsT-transformed $AdS_5 \times S^5$* , *JHEP* **07** (2008) 071 [[arXiv:0805.1070](#)] [[SPIRES](#)].
- [38] A. Catal-Ozer, *Lunin-Maldacena deformations with three parameters*, *JHEP* **02** (2006) 026 [[hep-th/0512290](#)] [[SPIRES](#)].
- [39] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for superconformal deformations of $N = 4$ super Yang-Mills theory*, *JHEP* **07** (2005) 045 [[hep-th/0503192](#)] [[SPIRES](#)].
- [40] N. Beisert and R. Roiban, *Beauty and the twist: the Bethe ansatz for twisted $N = 4$ SYM*, *JHEP* **08** (2005) 039 [[hep-th/0505187](#)] [[SPIRES](#)].
- [41] B.-H. Lee, R.R. Nayak, K.L. Panigrahi and C. Park, *On the giant magnon and spike solutions for strings on $AdS_3 \times S^3$* , *JHEP* **06** (2008) 065 [[arXiv:0804.2923](#)] [[SPIRES](#)].
- [42] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys. B* **536** (1998) 199 [[hep-th/9807080](#)] [[SPIRES](#)].
- [43] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-Einstein metrics on $S^2 \times S^3$* , *Adv. Theor. Math. Phys.* **8** (2004) 711 [[hep-th/0403002](#)] [[SPIRES](#)].
- [44] M. Cvetič, H. Lü, D.N. Page and C.N. Pope, *New Einstein-Sasaki spaces in five and higher dimensions*, *Phys. Rev. Lett.* **95** (2005) 071101 [[hep-th/0504225](#)] [[SPIRES](#)].
- [45] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, *An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals*, *JHEP* **06** (2005) 064 [[hep-th/0411264](#)] [[SPIRES](#)].
- [46] S. Benvenuti and M. Kruczenski, *From Sasaki-Einstein spaces to quivers via BPS geodesics: $L^{p,q,r}$* , *JHEP* **04** (2006) 033 [[hep-th/0505206](#)] [[SPIRES](#)].
- [47] S. Franco et al., *Gauge theories from toric geometry and brane tilings*, *JHEP* **01** (2006) 128 [[hep-th/0505211](#)] [[SPIRES](#)].
- [48] A. Butti, D. Forcella and A. Zaffaroni, *The dual superconformal theory for $L^{(p,q,r)}$ manifolds*, *JHEP* **09** (2005) 018 [[hep-th/0505220](#)] [[SPIRES](#)].

- [49] N. Itzhaki, I.R. Klebanov and S. Mukhi, *PP wave limit and enhanced supersymmetry in gauge theories*, *JHEP* **03** (2002) 048 [[hep-th/0202153](#)] [[SPIRES](#)].
- [50] J. Gomis and H. Ooguri, *Penrose limit of $N = 1$ gauge theories*, *Nucl. Phys. B* **635** (2002) 106 [[hep-th/0202157](#)] [[SPIRES](#)].
- [51] L.A. Pando Zayas and J. Sonnenschein, *On Penrose limits and gauge theories*, *JHEP* **05** (2002) 010 [[hep-th/0202186](#)] [[SPIRES](#)].
- [52] S. Kuperstein, O. Mintkevich and J. Sonnenschein, *On the pp-wave limit and the BMN structure of new Sasaki-Einstein spaces*, *JHEP* **12** (2006) 059 [[hep-th/0609194](#)] [[SPIRES](#)].
- [53] S. Benvenuti and E. Tonni, *Near-flat space limit and Einstein manifolds*, *JHEP* **02** (2008) 022 [[arXiv:0707.1676](#)] [[SPIRES](#)].
- [54] S. Benvenuti and E. Tonni, *Giant magnons and spiky strings on the conifold*, *JHEP* **02** (2009) 041 [[arXiv:0811.0145](#)] [[SPIRES](#)].
- [55] N. Kim, *Multispin strings on $AdS_5 \times T^{(1,1)}$ and operators of $N = 1$ superconformal theory*, *Phys. Rev. D* **69** (2004) 126002 [[hep-th/0312113](#)] [[SPIRES](#)].
- [56] D. Giataganas, *Semi-classical strings in Sasaki-Einstein manifolds*, [arXiv:0904.3125](#) [[SPIRES](#)].
- [57] H. Dimov, M. Michalcik and R.C. Rashkov, *More giant magnons on $T^{1,1}$* , work in progress.
- [58] K. Pilch and N.P. Warner, *A new supersymmetric compactification of chiral IIB supergravity*, *Phys. Lett. B* **487** (2000) 22 [[hep-th/0002192](#)] [[SPIRES](#)].
- [59] N. Halmagyi, K. Pilch, C. Romelsberger and N.P. Warner, *Holographic duals of a family of $N = 1$ fixed points*, *JHEP* **08** (2006) 083 [[hep-th/0506206](#)] [[SPIRES](#)].
- [60] H. Dimov, V.G. Filev, R.C. Rashkov and K.S. Viswanathan, *Semiclassical quantization of rotating strings in Pilch- Warner geometry*, *Phys. Rev. D* **68** (2003) 066010 [[hep-th/0304035](#)] [[SPIRES](#)].